Multiresolution Analysis of a Class of Nonstationary Processes

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Abstract— **Processing nonstationary signals is an important and challenging problem. We focus on the class of nonstationary processes with stationary increments of an arbitrary order, and place them in a multiscale framework. Unlike other related studies, we concentrate on the discrete-time analysis and derive a number of new results in addition to placing the related existing ones in the same framework. We extend the study to various parametric models for which we derive the resulting multiresolution description. We show that wide-sense stationarity may be achieved by adequately selecting the analysis wavelet. After generalizing the study to wavelet packet analysis, we show that the latter possesses additional properties which are useful in the presence of other types of nonstationarities.**

Index Terms—**Wavelets, wavelet packets, nonstationarity, stationary increments, ARIMA.**

I. Introduction

GIVEN THE ubiquitous presence of nonstationarities in various physical processes, research interest in nonstationary signal processing has been constantly growing. Much of the existing theory in estimation and detection relies on the assumption of stationarity of the observed process. To apply this theory, researchers have commonly had to assume a slow variation of the latter and subsequently use an adaptive scheme to track the variations. To mitigate the many practical cases for which this assumption is invalid, one may adopt an alternative approach which consists of introducing a windowing transformation, thereby justifying a local stationarity and making use of classical techniques in estimation/detection.

The advent of multiscale analysis theory together with the nice properties of wavelets, provided a potential and a framework for an efficient analysis of nonstationary processes. A number of papers have addressed the topic of wavelet decomposition of random processes [1]–[4] and only a few have specifically addressed the nonstationarity issue [5]–[11]. Flandrin [5] first presented some fundamental results on the time-scale analysis of the fractional Brownian motion (fBm). Other subsequent works [6], [7] provided more insight into the statistical characterization of the wavelet coefficients of the fBm. Masry [8] has generalized these results to redundant and orthonormal wavelet decompositions of processes with stationary increments. Recently, Houdré and Cambanis [9], [10] have derived other fundamental results on the wavelet transform of stochastic processes with stationary increments of an arbitrary order. This class of processes is often used in time-series analysis in applied fields such as economics, hydrology, physics, and systems modeling. All these approaches were in a continuous time/scale domain and have, to a great extent, explained many previous experimental observations.

In this paper, we adopt a discrete-time domain approach and use digital filtering techniques which are familiar in the applied sciences. In addition to deriving and presenting the main results in [9], [10] in a more readily applicable form, we obtain new results on the properties of a multiresolution analysis of the class of nonstationary processes with stationary increments of an arbitrary order. In the case of interest, the degree of nonstationarity of a process bears, in some sense, information on the amount of memory in a process. The multiscale analysis of such processes unfolds this memory and as a result, progressively induces stationarity. This very property will allow us to parametrically model this representation after we appropriately model the nonstationarity. We thus discuss two types of nonstationary processes which are characterized by the presence of one or more poles on the unit circle: i) processes corresponding to poles at $z = 1$ in the z plane,[1] ii) processes corresponding to poles at $z = -1$. We will also show that stationarity may be achieved by appropriately choosing the analyzing wavelet. After generalizing the above results to a wavelet packet decomposition, we demonstrate that wavelet packets, with their properties, provide a powerful tool for the analysis of these nonstationary processes.

The paper is organized as follows: In Section II, we give some background relevant to the remainder of the paper. In Section III, we develop the properties of the wavelet decomposition of nonstationary processes with stationary increments. In Section IV, we establish some new results on the wavelet transformation of stationary and nonstationary parametric models. We extend the previous results to wavelet packets in Section V. We then provide some concluding remarks.

II. Background

A. Multiscale Analysis

Multiscale signal analysis has received much attention over the last five years [12]–[14]. This is a result of its simple

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[1] These models are referred to in the literature as Autoregressive Integrated Moving Average processes and will later be described in more detail.

implementation and its mathematical properties which put particular emphasis on local features.

Multiscale analysis is based on a finite energy function $\psi(t)(\in L^2(\mathbb{R}))$ which satisfies some condition [14] to ensure the invertibility of the transform (i.e., reconstruction of an analyzed signal). This function is usually referred to as a mother wavelet. Among the interesting characteristics of $\psi(t)$ are its local support in time (or space) and a fast decay of its transform in the dual or frequency space. The translates and dilates of this function defined as

$$\psi_{2^j,\theta}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-\theta}{2^j}\right) \quad (1)$$

can be used to decompose a random process $x(t)$ and obtain the following coefficients:

$$\tilde{\mathcal{W}}_{2^j}^{\theta} = \frac{1}{\sqrt{2^j}}\int_{-\infty}^{\infty} x(t)\psi^*\left(\frac{t-\theta}{2^j}\right) dt \quad (2)$$

where * stands for conjugation.[2] If $x(t)$ is a second-order process, the above integral exits w.p.1 and defines a second-order random variable provided that [10]

$$\int_{-\infty}^{\infty} \sqrt{E\{|x(t)|^2\}}\left|\psi\left(\frac{t-\theta}{2^j}\right)\right| dt < \infty.$$

Subject to some conditions, one can construct an orthonormal basis, by restricting $\theta = 2^j k, k \in \mathbb{Z}$ and in which the expansion coefficients of $x(t)$ result by decimating $\tilde{\mathcal{W}}_{2^j}^{k}$

$$\mathcal{W}_j^k = \tilde{\mathcal{W}}_{2^j}^{2^j k}. \quad (3)$$

These coefficients represent the details of the process at resolution 2^{-j}. Such an orthonormal basis may be built from a multiresolution analysis of $L^2(\mathbb{R})$, and in which case the approximation of the signal at resolution 2^{-j} can be similarly described by

$$\tilde{\mathcal{A}}_{2^j}^{\theta} = \frac{1}{\sqrt{2^j}}\int_{-\infty}^{\infty} x(t)\phi^*\left(\frac{t-\theta}{2^j}\right) dt \quad (4)$$

where $\phi(\cdot)$ is a scaling function and from which we can again obtain the orthonormal coefficients

$$\mathcal{A}_j^k = \tilde{\mathcal{A}}_{2^j}^{2^j k}. \quad (5)$$

In signal processing, the observed process is available as sampled data $x(n)$ which for practical reasons, are often considered to be the approximation sequence $\{\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$ at resolution level j_0 [16]. As a result, the aforementioned decomposition coefficients can then be efficiently and recursively computed by using a bank of (paraunitary) Quadrature Mirror Filters (QMF) [17] whose impulse responses $\{g_k\}_{k\in\mathbb{Z}}$ and $\{h_k\}_{k\in\mathbb{Z}}$ are, respectively, based on the wavelet and the scaling functions. It is worth recalling that the paraunitary property is a result of the unitarity of

$$M(\omega) = \frac{1}{\sqrt{2}}\begin{bmatrix} H(e^{j\omega}) & H(-e^{j\omega}) \\ G(e^{j\omega}) & G(-e^{j\omega}) \end{bmatrix} \quad (6)$$

$(\forall\omega \in [0, 2\pi])$, where $H(z)$ and $G(z)$ are the z-transforms of $\{h_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively. Due to the lowpass/highpass characteristics of the QMFs, we also have $H(1) = \sqrt{2}$ and $G(1) = 0$. In the remainder of this paper, we will assume that $\{h_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are the impulse responses of Finite Impulse Response (FIR) filters. Recall that these sequences must be of the same even length [14].

A property of the flatness of the frequency responses of these filters can be related to the number of vanishing moments of $\psi(t)$ defined as [18]

$$\left.\frac{d^p\hat{\psi}(\omega)}{d\omega^p}\right|_{\omega=0} = \int_{-\infty}^{\infty} t^p\psi(t)\, dt, \quad p \in \{1, \cdots, r\}. \quad (7)$$

These moments vanish if and only if

$$\left.\frac{d^pH(e^{i\omega})}{d\omega^p}\right|_{\omega=\pi} = \sum_{n=-\infty}^{\infty}(-1)^n n^p h_n = 0, \quad p \in \{1, \cdots, r\} \quad (8)$$

which means that $H(z)$ (resp., $G(z)$) possesses a zero of order $r+1$ in $z = -1$ (resp., $z = 1$). For simplicity sake, we will be referring to this property as the *r-vanishing* property [19]. Note that the r-vanishing property is a necessary condition for the r-regularity of the wavelet in the Hölder sense [20].

Throughout this paper, we consider wavelets which are obtained from a multiresolution analysis as described in [12]. This is also the case when we use redundant wavelet transforms such as (2).

B. Adaptive Analysis: Wavelet Packets

A wavelet basis which is adequate or even optimal for representing an observed process, may result in a less adequate or even poor representation of another process. The choice of a basis best matched to a given observed process is thus of paramount importance, if further analysis is required. This is the motivation behind the generalization of wavelets, namely *wavelet packets*. Adopting the notation of [21], we denote by $\{W_m(t), m \in \mathbb{N}\}$ the functions of $L^2(\mathbb{R})$ satisfying

$$\int_{-\infty}^{\infty} W_0(t)\, dt = 1 \quad (9)$$

$$W_{2m}(t) = \sqrt{2}\sum_{k=-\infty}^{\infty} h_{-k}^* W_m(2t-k) \quad (10)$$

$$W_{2m+1}(t) = \sqrt{2}\sum_{k=-\infty}^{\infty} g_{-k}^* W_m(2t-k) \quad (11)$$

where $\{h_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are as previously defined (i.e., impulse responses of the QMF's).

If for every $j \in \mathbb{Z}$, we define the vector space $\Omega_{j,m} = \text{Span}\{W_m(2^{-j}t-k), k \in \mathbb{Z}\}$, one can then show that

$$\Omega_{j,m} = \Omega_{j+1,2m} \overset{\perp}{\oplus} \Omega_{j+1,2m+1} \quad (12)$$

where $\overset{\perp}{\oplus}$ stands for a direct sum of orthogonal spaces. It can further be shown that $\{2^{-j/2}W_m(2^{-j}t-k), k \in \mathbb{Z}\}$ is an orthonormal basis of $\Omega_{j,m}$. As a direct result, if we denote by $\mathcal{P}$

[2]Note that, under some conditions [15], a redundant representation may be considered as a collection of orthonormal decompositions and thus results in [3] could be applied.

a partition[3] of $\mathbb{R}^+$ into intervals $I_{j,m} = [2^{-j}m, \cdots, 2^{-j}(m+1)[, j \in \mathbb{Z}$ and $m \in \mathbb{N}$, then

$$L^2(\mathbb{R}) = \bigoplus_{(j,m)/I_{j,m}\in\mathcal{P}}^{\perp} \Omega_{j,m} \tag{13}$$

where the symbol "/" stands for "*such that.*" This is equivalent to saying that $\{2^{-j/2}W_m(2^{-j}t-k), k \in \mathbb{Z}, (j,m)/I_{j,m} \in \mathcal{P}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Such a basis is called a wavelet packet. Note that the wavelet basis constructed from a multiresolution analysis of $L^2(\mathbb{R})$ is a special wavelet packet basis where the scaling function is $\phi(t) = W_0(t)$ and the mother wavelet is $\psi(t) = W_1(t)$.

The coefficients of a redundant wavelet packet decomposition of a process $x(t)$ of $L^2(\mathbb{R})$ can, as previously, be defined as

$$\begin{aligned}\tilde{C}^{\theta}_{2^j,m} &= \left\langle x(t), \frac{1}{2^{j/2}} W_m\left(\frac{t-\theta}{2^j}\right)\right\rangle \\ &= \frac{1}{2^{j/2}} \int_{-\infty}^{\infty} x(t) W_m^*\left(\frac{t-\theta}{2^j}\right) dt.\end{aligned} \tag{14}$$

An orthonormal (nonredundant) decomposition is also obtained by decimation, i.e., $C^k_{j,m} = \tilde{C}^{2^j k}_{2^j,m}$, for $k \in \mathbb{Z}$ and $(j,m)/I_{j,m} \in \mathcal{P}$. The partition $\mathcal{P}$ will vary for different specific choices of the wavelet packet basis. Note that the optimal basis choice is usually the result of a tree search based on some selected criterion [22] and is not of interest herein. It is clear from definitions (9)–(11) that the coefficients of a wavelet packet decomposition of a process can also be efficiently computed by using a multistage two-channel filter bank [21].

Equations (9)–(11) can be used to show that

$$\hat{W}_m(0) = \int_{-\infty}^{\infty} W_m(t)\, dt = 0, \qquad m \neq 0 \tag{15}$$

where $\hat{W}_m(\omega)$ denotes the Fourier transform of $W_m(t)$. The number r of vanishing moments of a wavelet, previously defined, may be used together with (10) and (11) to show that the pth-order derivatives of $\hat{W}_{2m}(\omega)$ and $\hat{W}_{2m+1}(\omega)$ at 0 are linear combinations of the lth-order derivates of $\hat{W}_m(\omega)$ at 0, for $l \in \{0, \cdots, p\}$. Therefore, by using (15), it is easy to show by induction that (8) is equivalent to

$$\left.\frac{d^p \hat{W}_m(\omega)}{d\omega^p}\right|_{\omega=0} = \int_{-\infty}^{\infty} t^p W_m(t)\, dt = 0, \quad p \in \{1, \cdots, r\} \tag{16}$$

$\forall m \in \mathbb{N}\backslash\{0\}$. In this case, we will say that we have an r-vanishing wavelet packet decomposition.

C. A Class of Nonstationary Processes

Stationarity of an observed process is an important notion since it enables one to associate to it a shift-invariant distribution (in time), which in turn simplifies analytical approaches in solving related problems (e.g., estimation, detection). Processes whose statistics vary in time are called nonstationary processes [23]. An important class of processes whose increments hold a special stationarity property is that of nonstationary *Processes with Stationary Increments (PSI).*

Definition 1: Two continuous time random processes $x(t)$ and $y(t)$ are said to be processes with (wide-sense) mutually stationary increments of order $D \in \mathbb{N}$, if $\forall(\tau,\tau') \in \mathbb{R}^2, \Delta^D x(t;\tau)$, and $\Delta^D y(t;\tau')$ are mutually stationary where

$$\Delta^D x(t;\tau) = \sum_{p=0}^{D} (-1)^p \binom{D}{p} x(t-p\tau). \tag{17}$$

Note that $\Delta^D x(t;\tau)$ and $\Delta^D y(t;\tau')$ are mutually stationary if their crosscorrelation $E\{\Delta^D x(t;\tau)\Delta^D y(u;\tau')\}$ only depends on $t-u$, for every (t,u). When the above definition holds for $y(t) = x(t)$, $x(t)$ is said to be a continuous time PSI of order D [24]. A well-known process which has stationary increments of order 1, is the fBm [25].

The class of processes above may be extended to discrete time processes if D is taken to be in $\mathbb{R}$. This extension can readily be used to define, as in [26], the discrete time equivalent of the *fractional Gaussian noise*, which is the derivative (in the sense of Schwartz distributions) of the fBm.

Definition 2: Two discrete-time random processes $\{x_n\}_{n\in Z}$ and $\{y_n\}_{n\in Z}$ are said to be processes with (wide-sense) mutually stationary increments of order $D \in \mathbb{R}$, if

1) $\forall(k,k') \in \mathbb{Z}^2, \{\Delta^D x(n;k)\}_{n\in\mathbb{Z}}$ and $\{\Delta^D y(n;k')\}_{n\in\mathbb{Z}}$ exist in the mean-square sense and are mutually stationary where

$$\Delta^D x(n;k) = x_n + \sum_{p=1}^{\infty} (-1)^p \cdot \frac{D(D-1)\cdots(D-p+1)}{p!} x_{n-kp} \tag{18}$$

which can be shown to be equivalent to

2) $\{\Delta^D x_n\}_{n\in\mathbb{Z}}$ and $\{\Delta^D y_n\}_{n\in\mathbb{Z}}$ exist in the mean-square sense and are mutually stationary where

$$\Delta^D x_n = (1-q^{-1})^D x_n. \tag{19}$$

The symbol q^{-1} denotes the time-delay operator $(q^{-1}x_n = x_{n-1})$. When $D \in \mathbb{N}$, (18) reduces to a finite summation as in (17). When the definition above holds for $y_n = x_n, \{x_n\}_{n\in\mathbb{Z}}$ is said to be a discrete time PSI of order D. In fact, it is established in Appendix I that PSI are stationary (under weak conditions) if $D < 1/2$. It is also shown, in the same appendix, that the important property of stationarity of the increments of order D is invariant for discrete time PSI under decimation by a power of 2.

A simple example of a nonstationary continuous time process with stationary increments of order $D \in \mathbb{N}$ is

$$x(t) = \sum_{k=0}^{D} \xi_k t^k \tag{20}$$

[3] Recall that a partition $\mathcal{P}$ of a set $\mathcal{B}$ is the set of nonempty disjoint subsets whose union is $\mathcal{B}$.

where $\{\xi_k\}_{0\leq k\leq D}$ are second-order random variables. Its discrete representation, which also satisfies the PSI properties is

$$x_n = \sum_{k=0}^{D} \xi_k n^k. \tag{21}$$

Some more general characterizations of PSI may be found in [27].

An interesting process, which is related to PSI, may be constructed by linearly combining two or more consecutive samples of a special nonstationary process. These consecutive samples exhibit stationarity when linearly combined (at some order D), and this class will be referred to as that of nonstationary *Processes with Stationary Jumps* (PSJ) of order D.

Definition 3: Two discrete-time random processes $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ are processes with (wide-sense) mutually stationary jumps of order $D \in \mathbb{R}$, if $\{\Delta_{-1}^D x_n\}_{n\in\mathbb{Z}}$ and $\{\Delta_{-1}^D y_n\}_{n\in\mathbb{Z}}$ exist in the mean-square sense and are mutually stationary with

$$\Delta_{-1}^D x_n = (1+q^{-1})^D x_n. \tag{22}$$

For the sake of convenience, we also denote $\Delta^D x_n$ by $\Delta_1^D x_n$, thereby putting our description of PSI and PSJ in a unified notational framework. When the above definition holds for $y_n = x_n$, $\{x_n\}_{n\in\mathbb{Z}}$ is said to be a discrete-time PSJ of order D. The stationarity and decimation effect on PSJ are discussed in Appendix I. An example which illustrates the above definition and provides an intuitive appeal is

$$x_n = \xi_0 + \xi_1(-1)^n n \tag{23}$$

where $\{\xi_k\}_{k=0,1}$ are uncorrelated second-order random variables.

For ease of analysis and immediate extension to applications, a more quantitative description of these nonstationary processes is obtained by fitting parametric models to their increments or jumps. The Auto-Regressive Moving Average (ARMA) parametric model is a commonly used model to describe a wide variety of stationary processes. The readily applicable representation of an ARMA$(K,L)\{x_n\}_{n\in\mathbb{Z}}$ is given by

$$\alpha(q)x_n = \beta(q)\epsilon_n \tag{24}$$

where $\alpha(q) = 1+\alpha_1 q^{-1}+\cdots+\alpha_K q^{-K}$ is the regression operator, $\beta(q) = 1+\beta_1 q^{-1}+\cdots+\beta_L q^{-L}$ is the moving average operator, and $\{\epsilon_n\}_{n\in\mathbb{Z}}$ is a zero-mean stationary white noise. Furthermore, the roots of the polynomial $\alpha(q)$ must lie inside the unit circle to ensure the stability of the model. It is also assumed that $\alpha_K \neq 0, \beta_L \neq 0$ and $\alpha(z)$ and $\beta(z)$ have no common zeros to avoid degenerate values of the model orders K and L.

We model the nonstationarity by fitting an ARMA(K,L) model to $\{\Delta_\eta^D x_n\}_{n\in\mathbb{Z}}, \eta=-1$, which in turn results in an ARIMA(K,D,L) (Auto-Regressive Integrated Moving Average) process for $\{x_n\}_{n\in\mathbb{Z}}$ [23]. Special cases of ARIMA processes such as ARI processes ($L=0$) and IMA processes ($K=0$) may also be used. Note that a random walk is an ARIMA$(0,1,0)$.

III. Stationarization Properties of Wavelet Decompositions

A. Redundant Wavelet Decomposition of Discrete-Time PSI

Multiscale representations of observed random processes have been useful in a number of applications. The ability of wavelets to adapt well to local features of a process was often invoked to contend with nonstationary signals, and only recently has there been a theoretical investigation of the stationarizing properties of wavelets [5], [9], [10].

In carrying out a multiscale analysis of PSI, we show that the vanishing moments property of the chosen wavelet plays a key role in one's ability to effectively overcome the nonstationarity limitation. The results are first stated in the discrete-time case by assuming that the approximation coefficients at some resolution level j_0 form a PSI sequence. This assumption is particularly motivated by the fact that the approximation coefficients of a continuous-time PSI are discrete-time PSI, as will be shown in Section III-B.

Proposition 1: Given a random sequence $\{\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$, which is a PSI of order $D \in \mathbb{R}$ for any $j_0 \in \mathbb{Z}$, the wavelet coefficients $\{\tilde{\mathcal{W}}_{2^{j_1}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ and $\{\tilde{\mathcal{W}}_{2^{j_2}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$, with $\min\{j_1,j_2\} > j_0$, resulting from an r-vanishing wavelet decomposition, have mutually stationary increments of order $D-r-1$ and are therefore mutually stationary if $r > D - 3/2$.

Proof: Calling upon the filter bank implementation of a wavelet decomposition and making use of the time-delay operator q^{-1} we have

$$\tilde{\mathcal{W}}_{2^j}^{k2^{j_0}} = G(q^{2^{j-j_0-1}}) \prod_{l=0}^{j-j_0-2} H(q^{2^l})\mathcal{A}_{j_0}^k, \qquad j > j_0. \tag{25}$$

Using the r-vanishing property of the wavelet, one can rewrite the transfer function $G(z)$ as

$$G(z) = (1-z^{-1})^{r+1}G_0(z) \tag{26}$$

where $G_0(z)$ is a polynomial in z^{-1}, and factorizes $G(z)$ to exhibit its multiple pole at $z=1$ in the above equation. This equation can be rewritten at any given resolution j as

$$\begin{aligned} G(z^{2^{j-j_0-1}}) &= (1-z^{-2^{j-j_0-1}})^{r+1}G_0(z^{2^{j-j_0-1}}) \\ &= (1-z^{-1})^{r+1}G_1(z) \end{aligned} \tag{27}$$

where $G_0(z)$ is as previously defined, and $G_1(z)$ is appropriately chosen to factor out the multiple-pole term. The above expression can in turn be used to rewrite (27) as

$$\tilde{\mathcal{W}}_{2^j}^{k2^{j_0}} = G_1(q) \prod_{l=0}^{j-j_0-2} H(q^{2^l})\Delta^{r+1}\mathcal{A}_{j_0}^k. \tag{28}$$

It is clear from (28) that the sequence $\{\Delta^{D-r-1}\tilde{\mathcal{W}}_{2^j}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ is only a filtered version of $\{\Delta^D\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$, which is a stationary process. This property implies the mutual stationarity of the increments of order D of the wavelet coefficients at resolution levels j_1 and j_2. Furthermore, since two PSI sequences of order D are mutually stationary when $D < 1/2$ (cf. Appendix I) then so are the sequences of wavelet coefficients when $D - r - 1 < 1/2$. ■

In a similar way, one can easily show that the approximation coefficients $\{\tilde{\mathcal{A}}_{2^{j_1}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ and $\{\tilde{\mathcal{A}}_{2^{j_2}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ have mutually stationary increments of order D, for $\min\{j_1,j_2\} > j_0$. This can be easily understood by noting that $\{\tilde{\mathcal{A}}_{2^j}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ is the output of a filter which has no zero at $z = 1$. It is also clear that, when D is an integer, the wavelet coefficients are mutually stationary if $r \geq D-1$.

B. Redundant Wavelet Decomposition of Continuous-Time PSI

While processes are discrete-time in most signal processing applications, most real processes are continuous-time. It is thus natural to extend some of the previous properties established for discrete-time processes to continuous-time processes.

The results from Proposition 1 may thus be rewritten in the following way:

Corollary 1: For any $j_0 \in \mathbb{Z}$ the wavelet coefficients $\{\tilde{\mathcal{W}}_{2^{j_1}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ and $\{\tilde{\mathcal{W}}_{2^{j_2}}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$, with $\min\{j_1,j_2\} > j_0$, of a continuous-time PSI $x(t)$ of order $D \in \mathbb{N}$, obtained with an r-vanishing wavelet decomposition, form random sequences which have mutually stationary increments of order $D-r-1$ if $r \leq D-2$ and which are mutually stationary if $r \geq D-1$.

Proof: The definition of the approximation coefficients leads to

$$\Delta^D \mathcal{A}_{j_0}^k = \sum_{p=0}^{D}(-1)^p\binom{D}{p} \cdot \int_{-\infty}^{\infty} x(t+k-p)\,\frac{1}{2^{j_0/2}}\phi^*\left(\frac{t}{2^{j_0}}\right)dt \quad (29)$$

$$= \int_{-\infty}^{\infty} \Delta^D x(t+k;1) \cdot \frac{1}{2^{j_0/2}}\phi^*\left(\frac{t}{2^{j_0}}\right)dt, \qquad k \in \mathbb{Z}. \quad (30)$$

By Fubini's theorem, we find that, for every $(k,l) \in \mathbb{Z}^2$

$$E\{\Delta^D \mathcal{A}_{j_0}^k (\Delta^D \mathcal{A}_{j_0}^l)^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{\Delta^D x}(t-u+k-l) \cdot \frac{1}{2^{j_0}}\phi^*\left(\frac{t}{2^{j_0}}\right)\phi\left(\frac{u}{2^{j_0}}\right)dt\,du \quad (31)$$

where $\gamma_{\Delta^D x}(\cdot)$ is the crosscorrelation between $\Delta^D x(t+k;1)$ and $\Delta^D x(u+l;1)$. Therefore, $\{\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$ has stationary increments of order D. The use of Proposition 1 ends the proof. ■

C. Orthonormal Wavelet Decomposition of PSI

Orthonormal decompositions are of interest in a variety of applications; the importance of studying their properties for PSI cannot be overemphasized. Recalling that an orthonormal decomposition is achieved by a simple decimation of the corresponding redundant one, we expect the previous properties to carry over with minimal modification.

Corollary 2: Under the assumptions of Proposition 1 (resp., Corollary 1), for each $j > j_0$ (resp., $j \in \mathbb{Z}$), the random sequence $\{\mathcal{W}_j^k\}_{k\in\mathbb{Z}}$ is an PSI of order $D-r-1$ and is therefore stationary if $r > D-3/2$ (resp., $r \geq D-1$).

Proof: Using the fact that, for $j > j_0$, $\mathcal{W}_j^k$ is obtained by decimating $\tilde{\mathcal{W}}_{2^j}^{k2^{j_0}}$ by a factor 2^{j-j_0}, and that the invariance of the incremental stationarity under decimation is proved in Appendix I, the stationarity of order $D-r-1$ of $\{\mathcal{W}_j^k\}_{k\in\mathbb{Z}}$ follows. The complete stationarity is straightforwardly obtained, for a proper choice of wavelets. For a continuous time process, we let j_0 tend to $-\infty$ to establish the same property. ■

Note that the mutual stationarity properties of the wavelet coefficients $\{\mathcal{W}_{j_1}^k\}_{k\in\mathbb{Z}}$ and $\{\mathcal{W}_{j_2}^k\}_{k\in\mathbb{Z}}$ corresponding to two different resolution levels j_1 and j_2 no longer holds (e.g., for $r > D-3/2$), since by using Proposition 1, one can show that $E\{\mathcal{W}_{j_1}^k(\mathcal{W}_{j_2}^l)^*\}$ is a function of $2^{j_1}k - 2^{j_2}l$. For each j, the approximation coefficients $\{\mathcal{A}_j^k\}_{k\in\mathbb{Z}}$, however, have stationary increments of order D.

IV. Parametric Modeling of Multiscale Processes

Parametric modeling, as discussed earlier, has been very useful in studying stationary processes (ARMA and its variations). These models can be extended to appropriately model the class of nonstationary PSI by the previously introduced ARIMA processes. It is clear that if $\{\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$ is an ARIMA, for any $j_0 \in \mathbb{Z}$, the sequence of coefficients $\{\tilde{\mathcal{W}}_{2^j}^{k2^{j_0}}\}_{k\in\mathbb{Z}}, j > j_0$ also is ARIMA. This is a direct result of the fact that the wavelet coefficients sequence is nothing but a FIR filtered version of the approximation sequence as can be seen from (28).

We, therefore, focus on the development of orthonormal wavelet representations of ARIMA processes. We show below that the property of vanishing moments of the wavelet used in the analysis is a determining factor in the evolution of the ARIMA model. Clearly this first requires the analysis of the decimation effect.

Lemma 1: If $\{x_n\}_{n\in\mathbb{Z}}$ is an ARIMA (K,D,L), $D \in \mathbb{N}$, the decimated sequence $\hat{x}_n = x_{2n}$ is an ARIMA $(K,D,\hat{L})$, with $\hat{L} \leq (K+L+D)/2$. Furthermore, the poles of the resulting model are the squares of the poles of the original model.

Proof: See Appendix II.

The above result allows us to establish the following properties of a multiscale analysis of an ARIMA process:

Proposition 2: Let an r-vanishing wavelet decomposition be implemented by a QMF filter bank with FIR analysis filters of length $P+1$. If for any $j_0 \in \mathbb{Z}$, $\{\mathcal{A}_{j_0}^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K,D,L_{j_0,0})$, $D \in \mathbb{N}$, then, for $j > j_0$, the approximation sequence $\{\mathcal{A}_j^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K,D,L_{j,0})$, with

$$L_{j,0} \leq L_j^M = (K+D+P)(1-2^{j_0-j}) + L_{j_0,0}2^{j_0-j}. \quad (32)$$

If $r \leq D-2$, the wavelet sequence $\{\mathcal{W}_j^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K, D-r-1, L_{j,1})$ with

$$L_{j,1} \leq L_j^M - r - 1 \quad (33)$$

and, if $r \geq D-1$, it reduces to an ARMA$(K, L_{j,1})$ process, with

$$L_{j,1} \leq L_j^M - D. \quad (34)$$

Proof: We first address the property of the approximation coefficients and proceed to prove it by induction. We assume that it is satisfied for a $j > j_0$, and show that this implies its validity for $j+1$. The sequence $\{\mathcal{A}_{j+1}^k\}_{k\in\mathbb{Z}}$ is obtained by decimating $\{\overline{\mathcal{A}}_{j+1}^k\}_{k\in\mathbb{Z}}$ where "*overbar*" indicates that the expression is undecimated. The z-transforms corresponding to the sequences of coefficients are $\mathcal{A}_{j+1}(z)$ and $\overline{\mathcal{A}}_{j+1}(z)$, and clearly imply

$$\overline{\mathcal{A}}_{j+1}(z) = H(z)\mathcal{A}_j(z). \tag{35}$$

Knowing that $\{\mathcal{A}_j^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K, D, L_{j,0})$, $\{\overline{\mathcal{A}}_{j+1}^k\}_{k\in\mathbb{Z}}$ is also an ARIMA $(K, D, \overline{L}_{j+1})$ with

$$\overline{L}_{j+1} = L_{j,0} + P.$$

Using Lemma 1, we conclude that $\{\mathcal{A}_{j+1}^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K, D, L_{j+1,0})$ with $L_{j+1,0} \le (\overline{L}_{j+1} + D + K)/2$. This proves that (32) being satisfied for index j also holds for index $j+1$. The validity of the relation for $j = 1$ can also be straightforwardly checked with the help of Lemma 1.

In a very similar way, we note that the sequence of wavelet coefficients $\{\mathcal{W}_{j+1}^k\}_{k\in\mathbb{Z}}$ is obtained by decimating $\{\overline{\mathcal{W}}_{j+1}^k\}_{k\in\mathbb{Z}}$. The latter can be expressed in terms of its z-transform as

$$\overline{\mathcal{W}}_{j+1}(z) = G(z)\mathcal{A}_j(z). \tag{36}$$

Using the fact that the highpass analysis filter has the same length as the lowpass filter and possesses a zero of order $r+1$ at $z = 1$, one concludes that when $r \le D-2$, $\{\overline{\mathcal{W}}_{j+1}^k\}_{k\in\mathbb{Z}}$ is an ARIMA $(K, D-r-1, \overline{L}_{j+1} - r - 1)$ and when $r \ge D-1$, it is an ARMA $(K, \overline{L}_{j+1} - D)$. Using Lemma 1 immediately yields (33) and (34). ∎

Note that the order of the AR part is not modified by the multiresolution analysis whereas the order of the MA part $L_{j,0}$ is upper-bounded by $\max\{L_{j_0,0}, K+D+P\}$. Asymptotically, as $j \to \infty$, one can easily conclude that

$$L_{j,0} \le K + P + D. \tag{37}$$

It is also interesting to note that a multiscale representation of an IMA process results in an IMA coefficient sequence, while that of an ARI process will generally lead to an ARIMA sequence.

V. Generalized Multiscale Analysis: Wavelet Packets

A. Stationarizing Properties

Wavelet packet analysis is a generalized approach to an adaptive analysis for an optimal representation of a process. Its effect on different classes of processes is therefore of great interest. Since a wavelet basis is a particular wavelet packet, it is natural to investigate a generalization of the previously established results for wavelets. It will be shown below that some of the properties of PSI and PSJ in a wavelet packet basis are similar to those in a wavelet basis, with additional degrees of freedom.

Proposition 3: Given a random sequence $\{\tilde{C}_{2^{j_0},0}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$, which is an PSI (resp., PSJ) of order $D \in \mathbb{R}$ for any $j_0 \in \mathbb{Z}$, the wavelet packet coefficients $\{\tilde{C}_{2^j,m}^{k2^{j_0}}\}_{k\in\mathbb{Z}}, j > j_0, 0 \le m \le 2^{j-j_0} - 1$, resulting from an r-vanishing wavelet packet decomposition, are PSI (resp., PSJ) of order

$$D_{j,m} = D - (r+1)\sum_{l=1}^{j-j_0} \zeta_{j,m}(l) \tag{38}$$

$$\left(\text{resp., } D_{j,m} = D - (r+1)(1 - \zeta_{1,m}(1) + \sum_{l=2}^{j-j_0} \zeta_{j,m}(l))\right) \tag{39}$$

where $\zeta_{j,m}(1) \cdots \zeta_{j,m}(j-j_0)$ is the binary representation of m with $j - j_0$ digits, i.e.

$$m = \sum_{l=1}^{j-j_0} \zeta_{j,m}(l) 2^{j-j_0-l}.$$

Proof: We will only focus on the analysis of PSJ of order D since that of PSI can similarly be performed as with wavelets in Section I. Using (10), (11), and (14), we can write down for every $j > j_0$

$$\tilde{C}_{2^{j+1},2m}^{k2^{j_0}} = H(q^{2^{j-j_0}})\,\tilde{C}_{2^j,m}^{k2^{j_0}} \tag{40}$$

$$\tilde{C}_{2^{j+1},2m+1}^{k2^{j_0}} = G(q^{2^{j-j_0}})\,\tilde{C}_{2^j,m}^{k2^{j_0}}. \tag{41}$$

We now proceed with the proof, as previously, by induction, first assuming that the proposition is valid for index j. Using (26), we find that

$$\begin{aligned} G(z^{2^{j-j_0}}) &= (1 - z^{-2^{j-j_0}})^{r+1} G_0(z^{2^{j-j_0}}) \\ &= (1 + z^{-1})^{r+1} G_2(z), \qquad j > j_0 \end{aligned} \tag{42}$$

where $G_2(z)$ is a FIR transfer function. Equation (41) can then be rewritten

$$\tilde{C}_{2^{j+1},2m+1}^{k2^{j_0}} = G_2(q)\Delta_{-1}^{r+1}\tilde{C}_{2^j,m}^{k2^{j_0}}. \tag{43}$$

This then shows that $\{\tilde{C}_{2^{j+1},2m}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ and $\{\tilde{C}_{2^{j+1},2m+1}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ are PSJ of respective orders $D_{j+1,2m} = D_{j,m}$ and $D_{j+1,2m+1} = D_{j,m} - r - 1$. By further noting that

$$\zeta_{j+1,2m}(j - j_0 + 1) = 0 \quad \zeta_{j+1,2m+1}(j - j_0 + 1) = 1 \tag{44}$$

we have

$$\zeta_{j+1,2m}(l) = \zeta_{j+1,2m+1}(l) = \zeta_{j,m}(l), \quad l \le j - j_0. \tag{45}$$

Note that, if $D_{j,m}$ satisfies (39), $D_{j+1,2m}$ and $D_{j+1,2m+1}$ also satisfy it.

To conclude the proof, we need to check that the proposition is satisfied for $j = j_0 + 1$. Using the fact that

$$H(z) = (1 + z^{-1})^{r+1} H_1(z) \tag{46}$$

where $H_1(z)$ is a FIR transfer function, we can write

$$\tilde{C}_{2^{j_0+1},0}^{k2^{j_0}} = H_1(q)\,\Delta_{-1}^{r+1}\tilde{C}_{2^{j_0},0}^{k2^{j_0}}. \tag{47}$$

It follows that the sequences $\{\tilde{C}_{2^{j_0+1},0}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ and $\{\tilde{C}_{2^{j_0+1},1}^{k2^{j_0}}\}_{k\in\mathbb{Z}}$ are PSJ of respective orders $D - r - 1$ and D. ∎

Remarks:

Some special cases of the above results are discussed in the following remarks:

- The sequence $\{\tilde{C}^{k2^{j_0}}_{2^j,m}\}_{k\in\mathbb{Z}}, j>j_0$, is stationary, if $D_{j,m}<1/2$.
- It follows that if the original process is a PSI (resp., PSJ) of order D, the sequence $\{\tilde{C}^{k2^{j_0}}_{2^j,m}\}_{k\in\mathbb{Z}}, j>j_0$ is stationary, if $r>D-3/2$ and $m\neq 0$ (resp., $m\neq 2^{j-j_0-1}$) and that it is a PSI (resp., PSJ) of order D, if $m=0$ (resp., $m=2^{j-j_0-1}$).
- In the above cases (with redundant decompositions) stationarity of the sequences of coefficients implies their mutual stationarity. This can be easily seen by noting that these signals can be expressed as filtered versions of the same stationary sequence (which is $\{\Delta^D_{-1}\tilde{C}^{k2^{j_0}}_{2^{j_0},0}\}_{k\in\mathbb{N}}$, in the case of PSJ of order D).

Using orthogonal wavelet packet representations induces the properties which are given below:

Corollary 3: Let an r-vanishing wavelet packet be characterized by a partition $\mathcal{P}$ of $\mathbb{R}^+$ in intervals $I_{j,m}, j\in\mathbb{Z}, m\in\mathbb{N}$. Given a PSI (resp., PSJ) $\{C^k_{j_0,0}\}_{k\in\mathbb{Z}}$, of order $D\in\mathbb{R}$ for any $j_0\in\mathbb{Z}$, the wavelet packet coefficient sequence $\{C^k_{j,m}\}_{k\in\mathbb{Z}}, j>j_0, 0\le m\le 2^{j-j_0}-1, I_{j,m}\in\mathcal{P}$ is a PSI of order $D_{j,m}$ given by (38) (resp., (39)).

Proof: This result is straightforwardly obtained from Proposition 3 by first noting that $\{C^k_{j,m}\}_{k\in\mathbb{Z}}$ is a decimated version of $\{\tilde{C}^{k2^{j_0}}_{2^j,m}\}_{k\in\mathbb{Z}}$ by a factor 2^{j-j_0} and in addition, by recalling that the decimation by a power of 2 of a PSI or a PSJ of some order is an PSI of the same order. ∎

It must be emphasized that, for $\mathcal{P}$ to be a partition of $\mathbb{R}^+$, we can find at most one interval $I_{j,m}$ such that $m=0$ or $m=2^{j-j_0-1}$. This means that the orthogonal wavelet packet decomposition does not contain more than one set of coefficients $\{C^k_{j,m}\}_{k\in\mathbb{Z}}$ which cannot be stationarized by a choice of a high enough value of the vanishing order r. Note also that, for PSI (resp., PSJ), a judicious choice of the wavelet packet would be obtained by using (12), for $m=0$ (resp., $m=2^{j-j_0-1}$), in an iterative way, and making $j\to\infty$. By so doing, we are able to completely stationarize a nonstationary process. The wavelet decomposition of PSI is a special case of this approach. It is thus clear that the mutual stationarity between scales, here, is no longer valid.

B. Parametric Modeling of Wavelet Packet Coefficients of Nonstationary Processes

The parametric models of nonstationary processes were previously shown to have interesting properties in their wavelet-based representation. Their extension to wavelet packet bases is naturally required if additional degrees of freedom are desired. The following analysis allows us to rewrite results in Proposition 2 as follows:

Proposition 4: Let an r-vanishing wavelet packet be characterized by a partition $\mathcal{P}$ of $\mathbb{R}^+$ in intervals $I_{j,m}, j\in\mathbb{Z}, m\in\mathbb{N}$. Given a random sequence $\{C^k_{j_0,0}\}_{k\in\mathbb{Z}}$, such that $\{\Delta^D_\eta C^k_{j_0,0}\}_{k\in\mathbb{Z}}, D\in\mathbb{N}, \eta\in\{-1,1\}$, is an ARMA $(K, L_{j_0,0})$, the sequence of wavelet packet coefficients

$$\{C^k_{j,m}\}_{k\in\mathbb{Z}}, j>j_0, \quad 0\le m\le 2^{j-j_0}-1,\ I_{j,m}\in\mathcal{P}$$

obtained by a filter bank of length $P+1$, is an ARIMA $(K, D'_{j,m}, L_{j,m})$ where

$$D'_{j,m}=\max\{D_{j,m},0\} \tag{48}$$

with $D_{j,m}$ given by (38) (resp., (39)), and

$$L_{j,m}\le K+D'_{j,m}+P-(K+D+P-L_{j_0,0})2^{j_0-j}. \tag{49}$$

Proof: Using the results of Appendix II, and following an approach very similar to that used for Proposition 2, one concludes that, for $j>j_0$ and $m\in\{0,\cdots,2^{j-j_0}-1\}$, $\{C^k_{j,m}\}_{k\in\mathbb{Z}}$ is an ARIMA $(K, D'_{j,m}, L_{j,m})$ where $D'_{j,m}$ satisfies (48) and $L_{j,m}$ is such that

$$L_{j+1,2m}\le\frac{K+L_{j,m}+P+D'_{j,m}}{2}. \tag{50}$$

Similarly, we find that

$$L_{j+1,2m+1}\le\frac{K+\overline{L}_{j+1,2m+1}+\overline{D}_{j+1,2m+1}}{2} \tag{51}$$

where, if $r\le D'_{j,m}-2$

$$\overline{D}_{j+1,2m+1}=D'_{j,m}-r-1 \tag{52}$$

$$\overline{L}_{j+1,2m+1}\le L_{j,m}+P-r-1 \tag{53}$$

and, if $r\ge D'_{j,m}-1$

$$\overline{D}_{j+1,2m+1}=0 \tag{54}$$

$$\overline{L}_{j+1,2m+1}\le L_{j,m}+P-D'_{j,m}. \tag{55}$$

We can then rewrite (51)

$$L_{j+1,2m+1}\le\frac{K+L_{j,m}+P+D'_{j,m}}{2}-r-1, \quad \text{if } r\le D'_{j,m}-2 \tag{56}$$

$$L_{j+1,2m+1}\le\frac{K+L_{j,m}+P-D'_{j,m}}{2}, \quad \text{if } r\ge D'_{j,m}-1. \tag{57}$$

Since $D'_{j+1,2m}=D'_{j,m}$ and

$$D'_{j+1,2m+1}=\max\{D'_{j,m}-r-1,0\}$$

for $j>j_0$, we deduce the following:

$$L_{j+1,2m}-D'_{j+1,2m}\le(K+L_{j,m}-D'_{j,m}+P)/2$$

$$L_{j+1,2m+1}-D'_{j+1,2m+1}\le(K+L_{j,m}-D'_{j,m}+P)/2.$$

Then, for $j\ge j_0$, $L_{j,m}-D'_{j,m}$ is upper-bounded by a quantity Z_j which is independent of m

$$Z_{j+1}=\frac{K+Z_j+P}{2} \qquad Z_{j_0}=L_{j_0,0}-D. \tag{58}$$

Using the above recursion, one can reexpress Z_j as

$$Z_j=K+P-\frac{K+D+P-L_{j_0,0}}{2^{j-j_0}}, \qquad j\ge j_0 \tag{59}$$

which yields (49). ∎

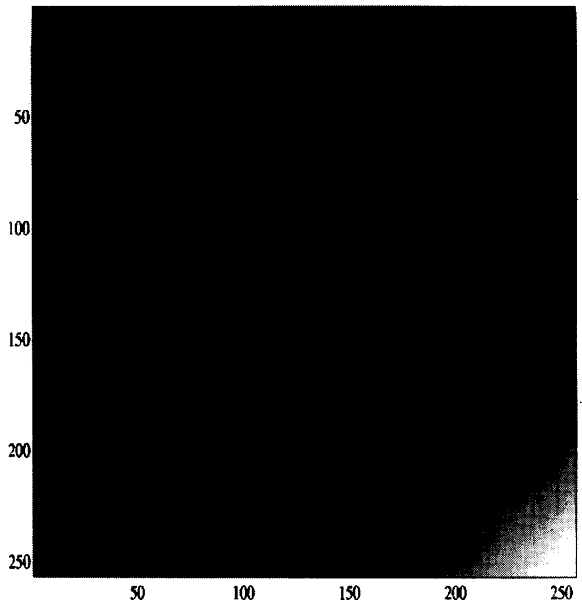

Fig. 1. Autocorrelation of the ARIMA process defined by (60).

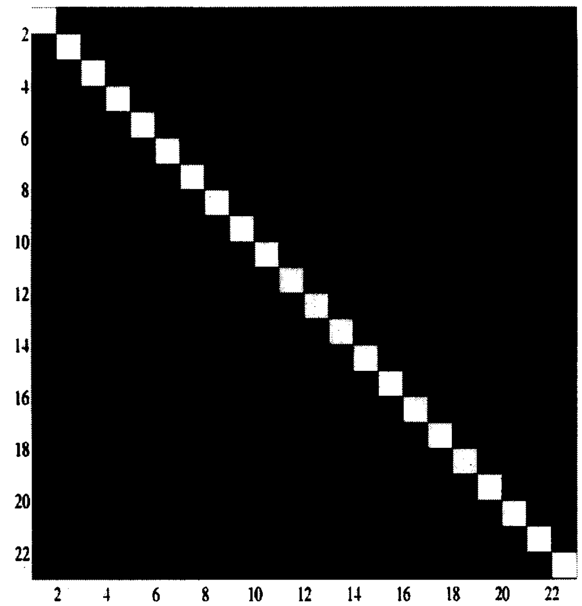

Fig. 2. Autocorrelation of the wavelet coefficients of the ARIMA process defined by (60) at resolution level 3.

Remark:

Note that, for $j > j_0$ and $m \in \{0, \cdots, 2^{j-j_0} - 1\}$ we have

$$L_{j,m} \leq \max\{L_{j_0,0}, K + D + P\} + D'_{j,m} - D$$

and that

$$\lim_{j \to \infty} L_{j,m} - D'_{j,m} \leq K + P.$$

VI. Examples

Example 1: Let $\{x_n\}_{n \in \mathbb{Z}}$ be an ARIMA $(1, 2, 1)$ such that

$$(1 - \rho q^{-1})\Delta^2 x_n = (1 + q^{-1})\epsilon_n \tag{60}$$

where $\rho = 0.8$ and $\{\epsilon_n\}_{n \in \mathbb{Z}}$ is an i.i.d. $N(0, 1/4)$. Fig. 1 depicts the correlation field $\gamma_x(n, p) = E\{x_n x_p\}$ of this process. This was the result of an ensemble average of 5000 realizations of a process of 256 samples. The highest gray-scales values of the image (bright pixels) correspond to the largest values of the autocorrelation function. For a 4-vanishing Daubechies wavelet [14] analysis, corresponding to QMF filters of length 10, we compute the autocorrelation matrices of the wavelet and approximation coefficients. The former is shown at resolution level $j = 3$ in Fig. 2, while the latter is displayed at the same resolution in Fig. 3. In the simulations, we assume $j_0 = 0$. The stationarity (resp., nonstationarity) of the wavelet (resp., approximation) coefficients is clearly evidenced by the equality (resp., in equality) of the components on the main and subdiagonals of the correlation matrix, i.e., Toeplitz structure (resp., non-Toeplitz) structure of the autocorrelation field. The ARIMA coefficients of the wavelet and approximation coefficients could also be computed using Lemma 1 and Proposition 2 as a guide.

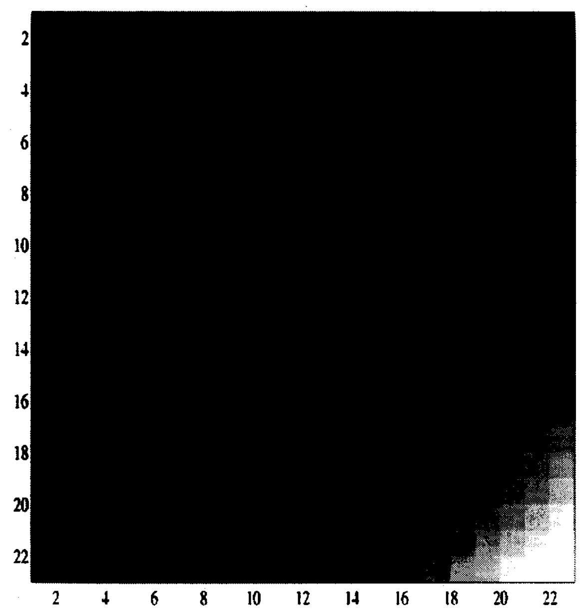

Fig. 3. Autocorrelation of the approximation coefficients of the ARIMA process defined by (60) at resolution level 3.

Example 2: A study case similar to that in Example 1, is performed for a PSJ $\{x_n\}_{n \in \mathbb{Z}}$ given by

$$(1 - 2\rho' \cos(2\pi\nu) q^{-1} + \rho'^2 q^{-2})\Delta_{-1} x_n = \epsilon_n \tag{61}$$

where $\rho' = 0.9, \nu = 0.3$, and $\{\epsilon_n\}_{n \in \mathbb{Z}}$ is an i.i.d. $N(0, 1)$, and for which a wavelet packet decomposition is carried out. A realization of the above process is shown in Fig. 4. The stationarity of the wavelet packet coefficients (when

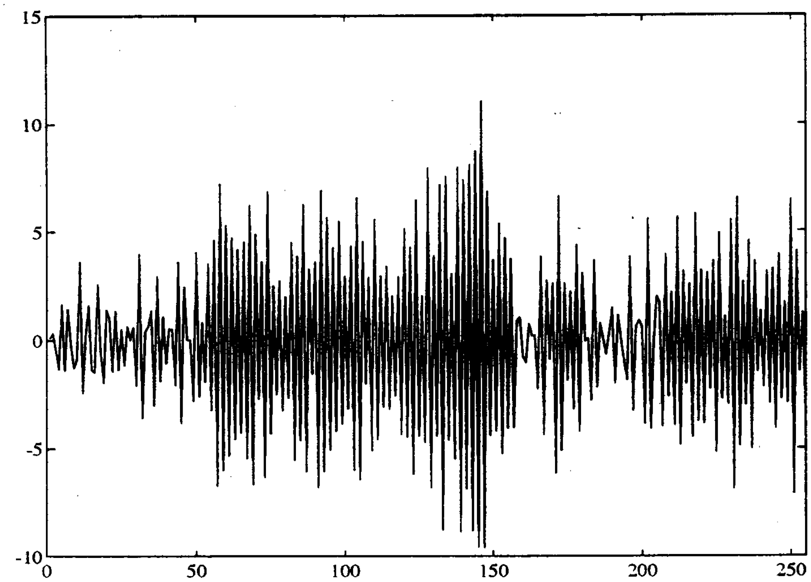

Fig. 4. Realization of the PSJ process defined by (61).

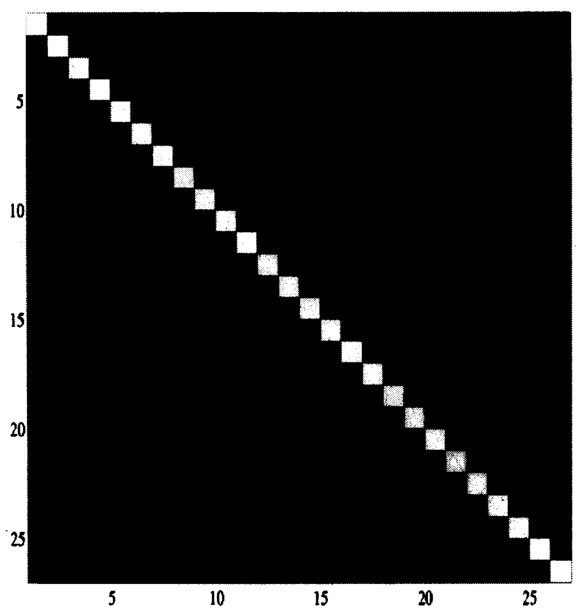

Fig. 5. Autocorrelation of the wavelet packet coefficients $j = 3, m = 5$ of the PSJ process defined by (61).

$m \neq 2^{j-1}$) is illustrated in Fig. 5 in a similar manner as that for the expressions in Example 1.

VII. Conclusions

We have given a multiscale discrete-time framework in which we can readily obtain some existing results on nonstationary processes with stationary increments. This in addition, has allowed us to derive a number of new results for related parametric models. We have extended the study to generalized wavelet packet analysis and have proposed another related class of nonstationary processes with nonstationary jumps. Applications are currently being investigated, and we expect these results to be useful in many other physical problems where analysis/synthesis of nonstationary processes are called for.

Appendix I
Statistical Properties of PSI and PSJ

A. PSI with $D < 1/2$

We follow a similar approach to that in [28] to provide details of the proof on the stationarity of PSI and PSJ of order $D < 1/2$. For this purpose we consider two processes $\{x_n\}_{n \in \mathbf{Z}}$ and $\{y_n\}_{n \in \mathbf{Z}}$ with mutual stationary increments or jumps of order D and denote, respectively, $\{\Delta_\eta^D x_n\}_{n \in \mathbf{Z}}$ and $\{\Delta_\eta^D y_n\}_{n \in \mathbf{Z}}, \eta \in \{-1, 1\}$ by $\{u_n\}_{n \in \mathbf{Z}}$ and $\{v_n\}_{n \in Z}$. We proceed to show that stationarity is achieved if $D < 1/2$ and the crosscorrelation $\gamma_{u,v}(k)$ is such that

$$\sum_{k=-\infty}^{\infty} |\gamma_{u,v}(k)|$$

is finite. By formally writing

$$x_n = (1 - \eta q^{-1})^{-D} u_n = \sum_{p=-\infty}^{\infty} a_p u_{n-p}, \tag{62}$$

$$a_p = \begin{cases} 0, & \text{if } p < 0 \\ 1, & \text{if } p = 0 \\ \eta^p \dfrac{D \cdots (D + p - 1)}{p!}, & \text{if } p \geq 1 \end{cases} \tag{63}$$

(and similarly for y_n), we find that

$$E\{x_n y_{n-k}^*\} = \sum_{l=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_p a_{p+l-k} \gamma_{u,v}(l). \tag{64}$$

The convergence of the series on the right-hand member of (64) is tantamount to establishing the mutual stationarity of $\{x_n\}_{n\in Z}$ and $\{y_n\}_{n\in Z}$. If we consider the absolute convergence of this series, we can rewrite (64) as

$$\sum_{l=-\infty}^{\infty}\left|\sum_{p=-\infty}^{\infty} a_p a_{p+l-k}\gamma_{u,v}(l)\right| \leq \sum_{p=-\infty}^{\infty} a_p^2 \sum_{l=-\infty}^{\infty} |\gamma_{u,v}(l)|. \quad (65)$$

It is then sufficient to prove that

$$\sum_{p=-\infty}^{\infty} a_p^2 = \sum_{p=0}^{\infty} a_p^2$$

is finite to conclude stationarity. For that purpose, we use Stirling's formula [28] to show that $a_p \sim \eta^p p^{D-1}/\Gamma(D)$, when $p \to \infty$, and conclude that the series is convergent if $D < 1/2$. ■

B. Decimation of PSI and PSJ

We investigate in this section the effect of decimation of PSI and PSJ. We show below that the decimation of PSI and PSJ of order D results in PSI of the same order. We designate a PSI or PSJ by $\{x_n\}_{n\in\mathbf{Z}}$ and let $\hat{x}_n = x_{2n}$ be the post-decimation (by a factor 2) sequence

$$\Delta^D \hat{x}_n = \Delta^D x(2n;2) \quad (66)$$

where

$$\begin{aligned}\Delta^D x(n;2) &= (1-q^{-2})^D x_n \\ &= (1+\eta q^{-1})^D \Delta_\eta^D x_n.\end{aligned} \quad (67)$$

The sequence $\{\Delta^D \hat{x}_n\}_{n\in\mathbf{Z}}$ is clearly stationary as it is a result of decimating $\{\Delta^D x(n;2)\}_{n\in\mathbf{Z}}$, which is stationary by definition. This result is straightforwardly extended to a decimation by any power of 2 factor.

Appendix II
Decimation of Nonstationary Parametric Models

In this appendix, we prove that if x_n is a nonstationary process such that $\Delta_\eta^D x_n, D \in \mathbb{N}, \eta \in \{-1, 1\}$, is an ARMA$(K, L)$, the decimated sequence $\hat{x}_n = x_{2n}$ is an ARIMA$(K, D, \hat{L})$, with $\hat{L} \leq (K + L + D)/2$. Furthermore, we show that the poles of the resulting model are the squares of the poles of the original model.

Let us first assume that the original signal $\{x_n\}_{n\in\mathbf{Z}}$ is an ARMA process ($D = 0$) defined by

$$x_n = \frac{\beta(q)}{\alpha(q)}\epsilon_n \quad (68)$$

where $\alpha(q)$ and $\beta(q)$ designate the AR and MA parts of respective orders K and L and $\{\epsilon_n\}_{n\in\mathbf{Z}}$ is the input noise of variance σ^2. The Power Spectral Density (PSD) of this process is then

$$S_x(\omega) = \sigma^2 \frac{|\beta(e^{i\omega})|^2}{|\alpha(e^{i\omega})|^2}. \quad (69)$$

After the decimation, we obtain a stationary process $\{\hat{x}_n\}_{n\in\mathbf{Z}}$ whose autocorrelation is obtained by decimating the autocorrelation of $\{x_n\}_{n\in\mathbf{Z}}$ by a factor 2. So, its PSD is

$$S_{\hat{x}}(\omega) = \frac{1}{2}\left[S_x\left(\frac{\omega}{2}\right) + S_x\left(\frac{\omega}{2} + \pi\right)\right]. \quad (70)$$

The above equation may be rewritten

$$S_{\hat{x}}(\omega) = \sigma^2 \frac{|\hat{\beta}(e^{i\omega})|^2}{|\hat{\alpha}(e^{i\omega})|^2} \quad (71)$$

where

$$\hat{\alpha}(e^{i\omega}) = \alpha(e^{i(\omega/2)})\alpha(-e^{i(\omega/2)}), \quad (72)$$

$$\begin{aligned}|\hat{\beta}(e^{i\omega})|^2 = \tfrac{1}{2}[&|\alpha(e^{i(\omega/2)})\beta(-e^{i(\omega/2)})|^2 \\ &+ |\alpha(-e^{i(\omega/2)})\beta(e^{i(\omega/2)})|^2].\end{aligned} \quad (73)$$

The function $\hat{\alpha}(e^{i2\omega})$ (resp., $|\hat{\beta}(e^{i2\omega})|^2$) is an even polynomial (resp., noncausal polynomial) in $e^{i\omega}$ and, consequently, $\hat{\alpha}(e^{i\omega})$ (resp., $|\hat{\beta}(e^{i\omega})|^2$) is a polynomial (resp., noncausal polynomial) in $e^{i\omega}$. The PSD of $\{\hat{x}_n\}_{n\in\mathbf{Z}}$ being a rational fraction implies that the process is an ARMA. In addition, it can be easily shown that the poles of $\hat{\alpha}(e^{i\omega})$ are the squares of those poles of $\alpha(e^{i\omega})$ (poles of original model) and that they are consequently also inside the unit circle. Similarly, if K and L have the same parity (resp., different parities), one can check that the order[4] of $|\hat{\beta}(e^{i\omega})|^2$ is less than or equal to (resp., strictly less than) $(K + L)/2$. Therefore, the MA part of the model, which is obtained by a spectral factorization (using Riesz theorem) of (73), is a polynomial of the same order.

In the general case where $D \neq 0$, by using (66) and (67), the sequence $\{\Delta^D \hat{x}_n\}_{n\in\mathbf{Z}}$ is obtained by decimating $\{\Delta^D x(n;2)\}_{n\in\mathbf{Z}}$, assumed to be an ARMA process. The above results may be used together with (67) to show that the order of the MA part of the original process now becomes $L + D$. ■

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